

TOPICS IN COMPLEX ANALYSIS @ EPFL, FALL 2024
SOLUTION SKETCHES TO HOMEWORK 5

MATHIAS BRAUN AND WENHAO ZHAO

Homework 5.1 (Examples of infinite products of functions*). Examine for which $z \in \mathbb{C}$ the following products converge absolutely and determine the largest open set $U \subset \mathbb{C}$ on which they converge locally normally.

- a. $\prod_{n=1}^{\infty} (1 + z^n).$
- b. $\prod_{n=1}^{\infty} \left[1 + \frac{z^n}{n!} \right].$
- c. $\prod_{n=1}^{\infty} \cos(2^{-n}z).$

Homework 5.2 (Further practice on infinite products). Show the product

$$H(z) = z \prod_{n=1}^{\infty} \left(1 + \frac{z}{n} \right) e^{-z/n}$$

converges locally normally on $\mathbb{C}^1$.

Solution. As we shall see it suffices to bound the quantity $(1 - z) e^z - 1$ on $B_1(0)$. Using the series expansion of the exponential function we have

$$(1 - z) e^z - 1 = \sum_{k=1}^{\infty} \frac{z^k}{k!} - \sum_{k=0}^{\infty} \frac{z^{k+1}}{k!} = \sum_{k=1}^{\infty} \frac{z^k(1 - k)}{k!} = \sum_{k=2}^{\infty} \frac{z^k(1 - k)}{k!}.$$

Taking the modulus yields

$$|(1 - z) e^z - 1| \leq \sum_{k=2}^{\infty} \frac{|z|^k}{(k - 2)!} = |z|^2 e^{|z|}.$$

In particular, for every $z \in B_1(0)$ we have the (nonoptimal) estimate

$$|(1 - z) e^z - 1| \leq e |z|^2.$$

Next fix a compact set $K \subset \mathbb{C}$. Then there exists $n(K) \in \mathbb{N}$ such that for all $n \geq n(K)$ we have $-z/n \in B_1(0)$ for every $z \in K$. We conclude

$$\sum_{n \geq n(K)} \sup_{z \in K} \left| \left[1 + \frac{z}{n} \right] e^{-z/n} - 1 \right| \leq e \sup_{z \in K} |z|^2 \sum_{n \geq n(K)} \frac{1}{n^2} < \infty.$$

By definition this shows the locally normal convergence of the infinite product H .

Date: October 28, 2024.

¹One can show that the limit $\gamma := \lim_{n \rightarrow \infty} \sum_{k=1}^n 1/k - \log(n)$ exists (which is called *Euler–Mascheroni constant*). Then $\Gamma(z) := e^{-\gamma z}/H(z)$ yields an alternative representation of the Eulerian Γ -function for every $z \in \mathbb{C}$ with $\Re z > 0$.

Homework 5.3 (Sine product formula). The goal of this exercise is to derive the following product formula for every $z \in \mathbf{C}$:

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left[1 - \frac{z^2}{n^2} \right].$$

This will be done in several steps.

- a. Use the partial fraction decomposition of $\pi^2/\sin^2(\pi z)$ from Homework 3.2 to show the following identity for every $z \in \mathbf{C} \setminus \mathbf{Z}$, with a suitable notion of convergence for the series on the right hand side²:

$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}.$$

- b. For $n \in \mathbf{N}$, define the function $f_n: \mathbf{C} \rightarrow \mathbf{C}$ by $f_n(z) := 1 - z^2/n^2$. Compute the logarithmic derivative f'/f for the assignment $f(z) := \pi z \prod_{n=1}^{\infty} f_n(z)$.
c. Conclude by comparing suitable terms.

Solution. a. Recall $\cot(\pi z) = \cos(\pi z)/\sin(\pi z)$ for every $z \in \mathbf{C} \setminus \mathbf{Z}$, thus

$$\pi \cot(\pi z)' = -\pi^2 \frac{\sin^2(\pi z) + \cos^2(\pi z)}{\sin^2(\pi z)} = -\frac{\pi^2}{\sin^2(\pi z)}.$$

Inserting the result from Homework 3.2, we deduce every $z \in \mathbf{C} \setminus \mathbf{Z}$ obeys

$$\pi \cot(\pi z)' = -\sum_{n \in \mathbf{Z}} \frac{1}{(z-n)^2} = -\frac{1}{z^2} - \sum_{n \in \mathbf{N}} \frac{1}{(z-n)^2} + \frac{1}{(z+n)^2} = -\frac{1}{z^2} - 2 \sum_{n \in \mathbf{N}} \frac{z^2 + n^2}{(z^2 - n^2)^2}.$$

As shown in Homework 3.2, the right-hand side series converges locally uniformly (and locally normally, which allows us to rearrange terms as above) on $\mathbf{C} \setminus \mathbf{Z}$. As $\mathbf{C} \setminus \mathbf{Z}$ is a domain we can apply the hint to the sequence $(g_k)_{k \in \mathbf{N}}$, where the function $g_k: \mathbf{C} \setminus \mathbf{Z} \rightarrow \mathbf{C}$ is defined by the assignment

$$g_k(z) := \frac{1}{z} + 2 \sum_{n=1}^k \frac{z}{z^2 - n^2}.$$

However, it is quite difficult to evaluate the limit of $g_k(z_0)$ as $k \rightarrow \infty$ for $z_0 \in \mathbf{C} \setminus \mathbf{Z}$ as $k \rightarrow +\infty$. Hence we use the hint in an abstract way. Note that for $z \in \mathbf{C} \setminus \mathbf{Z}$, we have

$$g_k'(z) = -\frac{1}{z^2} + 2 \sum_{n=1}^k \frac{z^2 - n^2 - 2z^2}{(z^2 - n^2)^2} = -\frac{1}{z^2} - 2 \sum_{n=1}^k \frac{z^2 + n^2}{(z^2 - n^2)^2}.$$

In particular, we obtain $g_k' \rightarrow \pi \cot(\pi \cdot)'$ locally uniformly on $\mathbf{C} \setminus \mathbf{Z}$ as $k \rightarrow \infty$. Moreover, the sequence $(g_k(1/2))_{k \in \mathbf{N}}$ is bounded. Hence, by compactness there exists $c \in \mathbf{C}$ such that $g_k(1/2) \rightarrow c$ as $k \rightarrow \infty$ along a subsequence we do not relabel³. By the hint,

$$\lim_{k \rightarrow \infty} g_k = \underbrace{\pi \cot(\pi \cdot) + c - \pi \cot\left(\frac{\pi}{2}\right)}_{=: c_0}$$

locally uniformly on $\mathbf{C} \setminus \mathbf{Z}$, so that

$$z \cot(\pi z) = \frac{1}{z} + \sum_{n=1}^k \frac{2z}{z^2 - n^2} + c_0. \quad (5.1)$$

²**Hint.** You may use the following fact without proof. If $D \subset \mathbf{C}$ is a domain, $(f_n)_{n \in \mathbf{N}}$ is a sequence of continuously differentiable functions $f_n: D \rightarrow \mathbf{C}$, $f: D \rightarrow \mathbf{C}$ is continuously differentiable, $f_n(z_0) \rightarrow f(z_0)$ as $n \rightarrow \infty$ for some $z_0 \in D$, and $f_n' \rightarrow f'$ locally uniformly as $n \rightarrow \infty$, then $f_n \rightarrow f$ locally uniformly as $n \rightarrow \infty$.

³Later the conclusions will be the same for all subsequences. This will show convergence of the entire sequence, as discussed in the lecture.

We claim $c_0 = 0$. To this aim, we observe that $\pi \cot(\pi z)$ has a first-order pole in $z = 0$. Hence its Laurent series at the origin reads

$$\pi \cot(\pi z) = \sum_{j=-1}^{\infty} a_j z^j.$$

From (5.1) we obtain $a_{-1} = 1$ and $a_0 = c_0$. Another formula for a_0 is

$$a_0 = \frac{d}{dz} \Big|_0 z \pi \cot(\pi z) = \lim_{z \rightarrow 0} \pi \lim_{z \rightarrow 0} \left[\cot(\pi z) - \frac{\pi z}{\sin^2(\pi z)} \right] = \pi \lim_{z \rightarrow 0} \left[\frac{\cos(\pi z) \sin(\pi z) - \pi z}{\sin^2(\pi z)} \right].$$

Since $\cos(\pi z) = 1 + O(z^2)$ as $z \rightarrow 0$ and $\sin(\pi z) = \pi z + O(z^3)$ as $z \rightarrow 0$, we obtain $\cos(\pi z) \sin(\pi z) - \pi z = O(z^3)$ as $z \rightarrow 0$. The above equality yields $a_0 = c_0 = 0$ as claimed.

b. Note that for $z \in \mathbb{C} \setminus \{\pm n\}$ we have

$$\frac{f'_n(z)}{f_n(z)} = \frac{2z}{z^2 - n^2}$$

Next, we argue the product $f = \pi z \prod_{n=1}^{\infty} f_n$ converges locally normally on $\mathbb{C}$. Indeed, let $K \subset \mathbb{C}$ be compact. Then

$$\sum_{n=1}^{\infty} \sup_{z \in K} \left| \frac{z^2}{n^2} \right| \leq \sup_{z \in K} |z|^2 \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

Hence we can apply Proposition 3.12 from the lecture notes, yielding

$$\frac{f'(z)}{f(z)} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{f'_n(z)}{f_n(z)} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}$$

for every $z \in \mathbb{C} \setminus \mathbb{Z}$, where we used that $Z(f) = \mathbb{Z}$ by Lemma 3.11.

c. Note that the logarithmic derivative of the assignment $h(z) := \sin(\pi z)$ on $\mathbb{C} \setminus \mathbb{Z}$ is given by $\pi \cot(\pi z)$. Hence a. and b. imply

$$\frac{h'}{h} = \frac{f'}{f}$$

on the domain $\mathbb{C} \setminus \mathbb{Z}$. In particular, on this set we have

$$\left[\frac{h}{f} \right]' = \frac{h'f - hf'}{f^2} = 0.$$

Thus there exists a constant $\hat{c} \in \mathbb{C}$ such that $h = \hat{c}f$ on $\mathbb{C} \setminus \mathbb{Z}$ and by the identity theorem this extends to the entire complex plane. In order to conclude we have to show $\hat{c} = 1$. To this aim, note that whenever $z \neq 0$ we can write

$$\frac{\sin(\pi z)}{\pi z} = \hat{c} \prod_{n=1}^{\infty} \left[1 - \frac{z^2}{n^2} \right].$$

Letting $z \rightarrow 0$ we deduce $\hat{c} = 1$, which finishes the proof.

Homework 5.4 (Consequences of the sine product formula). Use the product formula from the previous exercise to show the following statements.

a. $\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{4n^2}{4n^2 - 1}.$

b. $\cos(\pi z) = \prod_{n=1}^{\infty} \left[1 - \frac{4z^2}{(2n-1)^2} \right].$

c. $\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$ ⁴

⁴**Hint.** Use a Taylor expansion.

Solution. a. We insert $z = 1/2$ into the sine product formula

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left[1 - \frac{z^2}{n^2} \right]$$

and obtain

$$1 = \frac{\pi}{2} \prod_{n=1}^{\infty} \frac{4n^2 - 1}{4n^2}.$$

Taking the inverse of the infinite product (which is allowed term by term as the limit is not zero) we deduce the claimed formula.

b. Note that by the double angle formula, every $z \in \mathbf{C}$ satisfies $\sin(2z) = 2 \sin(z) \cos(z)$. (This is well-known for $z \in \mathbf{R}$ and extends to $z \in \mathbf{C}$ by the identity theorem. Thus,

$$\begin{aligned} \sin(\pi z) \cos(\pi z) &= \frac{1}{2} 2\pi z \prod_{n=1}^{\infty} \left[1 - \frac{4z^2}{n^2} \right] \\ &= \pi z \prod_{n=1}^{\infty} \left[1 - \frac{4z^2}{(2n)^2} \right] \prod_{n=1}^{\infty} \left[1 - \frac{4z^2}{(2n-1)^2} \right] \\ &= \sin(\pi z) \prod_{n=1}^{\infty} \left[1 - \frac{4z^2}{(2n-1)^2} \right]. \end{aligned}$$

The claim follows from the identity theorem.

c. Let us rewrite the product formula as a Taylor series. Locally uniformly on $\mathbf{C}$,

$$\begin{aligned} \lim_{m \rightarrow \infty} \sum_{n=1}^m (-1)^{n-1} \frac{(\pi z)^{2n-1}}{(2n-1)!} &= \sin(\pi z) \\ &= \lim_{m \rightarrow +\infty} \pi z \prod_{n=1}^m \left[1 - \frac{z^2}{n^2} \right] \\ &= \pi z - \lim_{m \rightarrow +\infty} \left[\pi z \sum_{n=1}^m \frac{z^2}{n^2} + g_m(z) \right], \end{aligned}$$

where $g_m(z)$ is the remainder. Since the series $\sum_{n=1}^m z^2/n^2$ converges locally uniformly, also g_m converges locally uniformly on $\mathbf{C}$ to some function $g: \mathbf{C} \rightarrow \mathbf{C}$. Moreover, each g_m has a zero of order 5 in zero, so that $g_m^{(k)}(0) = 0$ for all $0 \leq k \leq 4$. From Theorem 1.5 in the lecture notes we deduce g has a zero of order at least 5 in zero. Hence we can write

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(\pi z)^{2n-1}}{(2n-1)!} = \pi z - \pi z^3 \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=5}^{\infty} a_n z^n.$$

Comparing the coefficient of z^3 yields the claim.